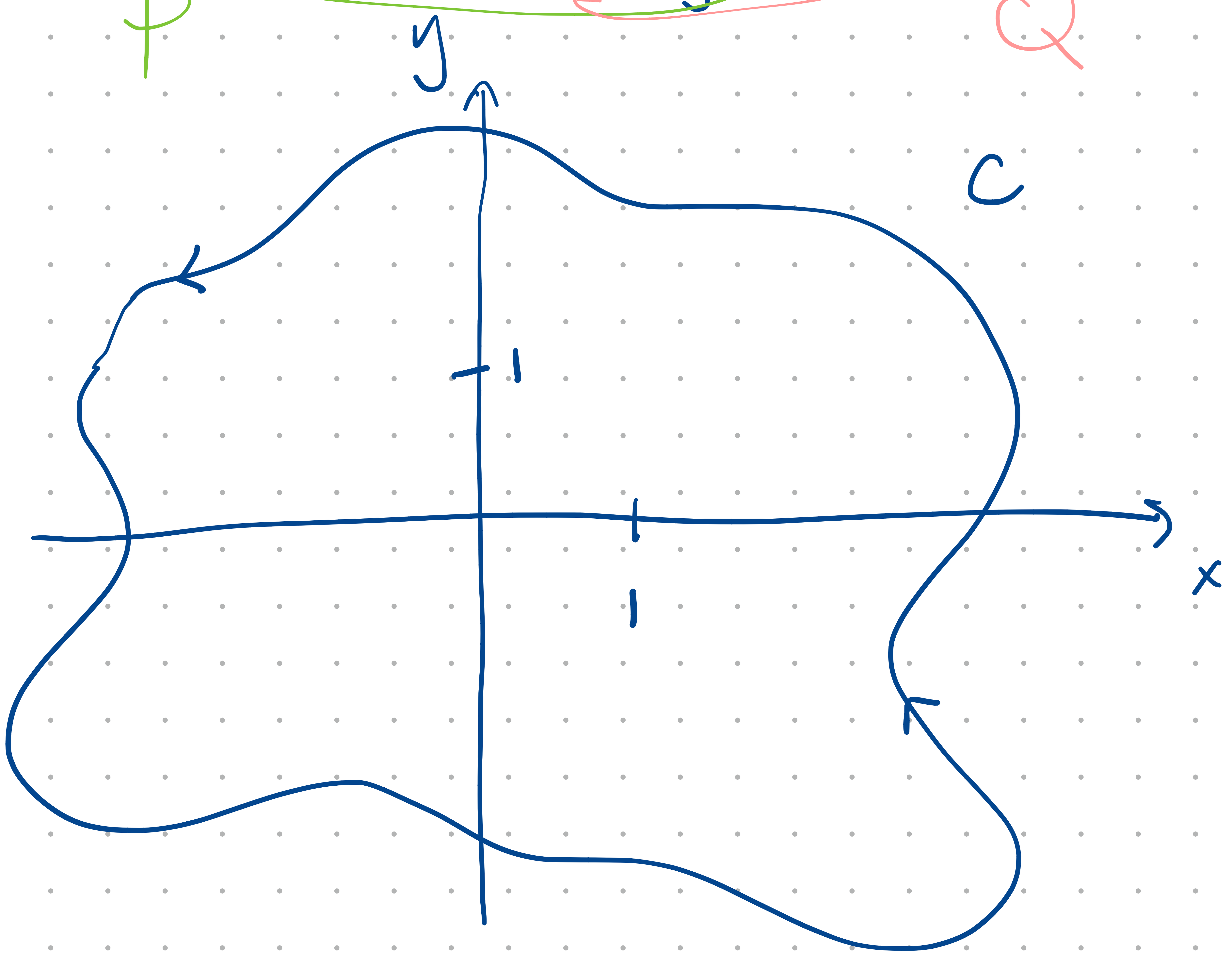
Stewart p. 1150 #38

$$\vec{F} = \frac{(2x^3+2xy^2-2y)\hat{\imath} + (2y^3+2x^2y+2x)\hat{\jmath}}{x^2+y^2}$$

P

Q



What is $\oint_C \vec{F} \cdot d\vec{r}$?

First thought: this problem looks impossible, b/c we don't know explicit description of C.

Second thought Maybe that's b/c it doesn't matter; i.e. maybe $\vec{F}$ is conservative

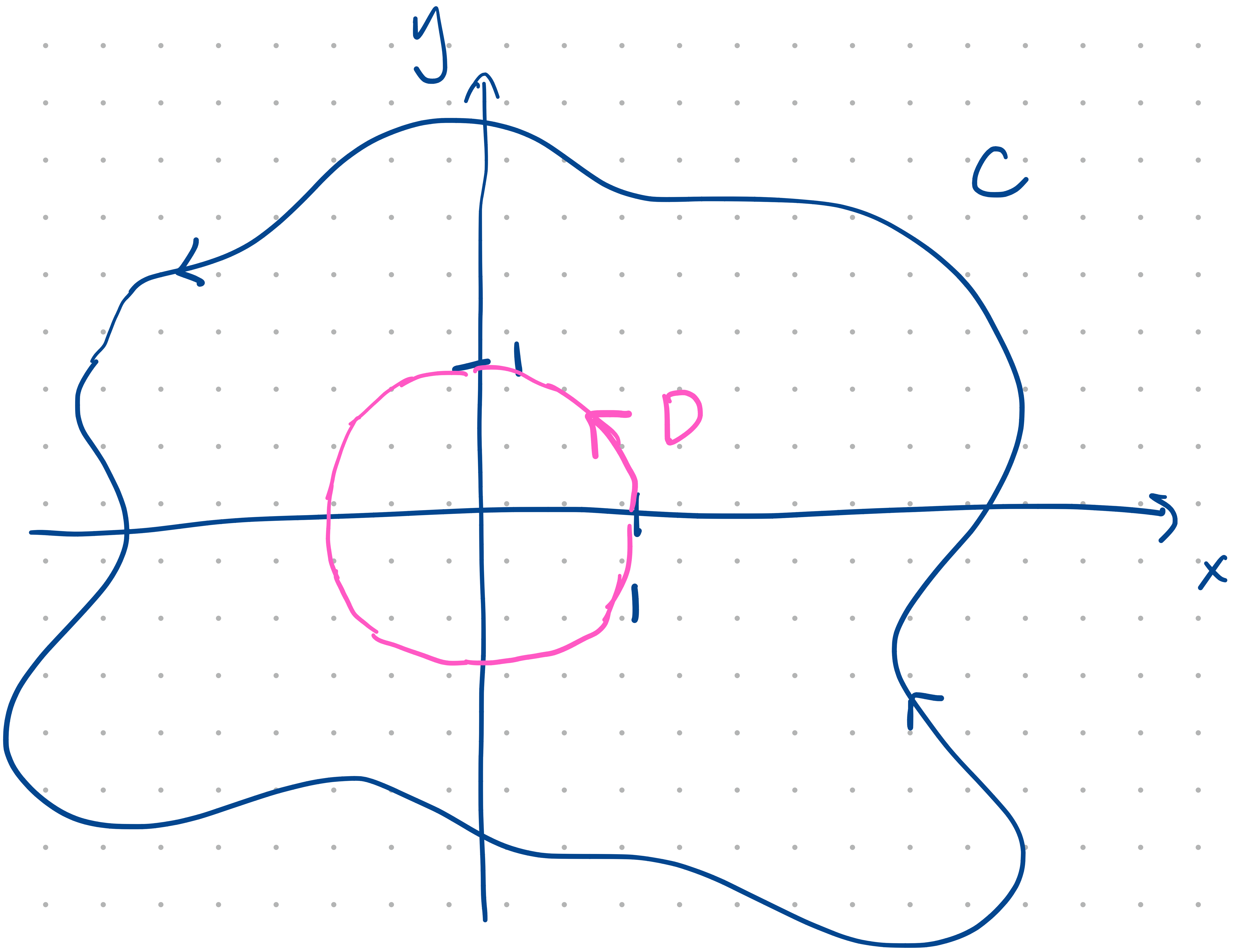
(so $\oint_C \vec{F} \cdot d\vec{r} = 0$).

In line w/ this second thought, we compute $Q_x - P_y$ and we find

$$Q_x - P_y = 0 \quad \text{(check!!)}$$

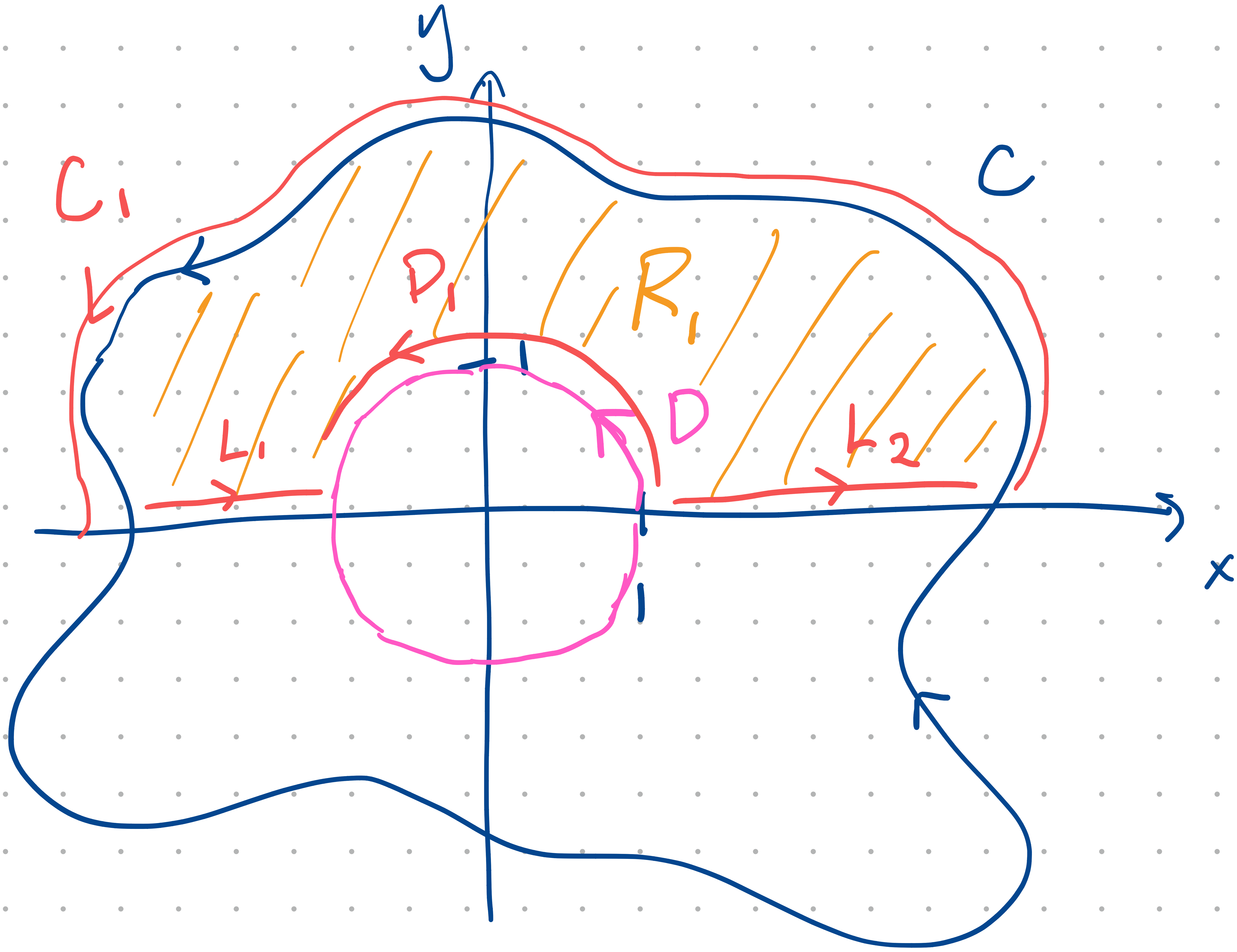
⚠ But we can't conclude $\vec{F}$ is conservative, nor can we apply Green's Thm to $\oint_C \vec{F} \cdot d\vec{r}$ b/c $\vec{F}$ is not defined @ $(0,0)$.

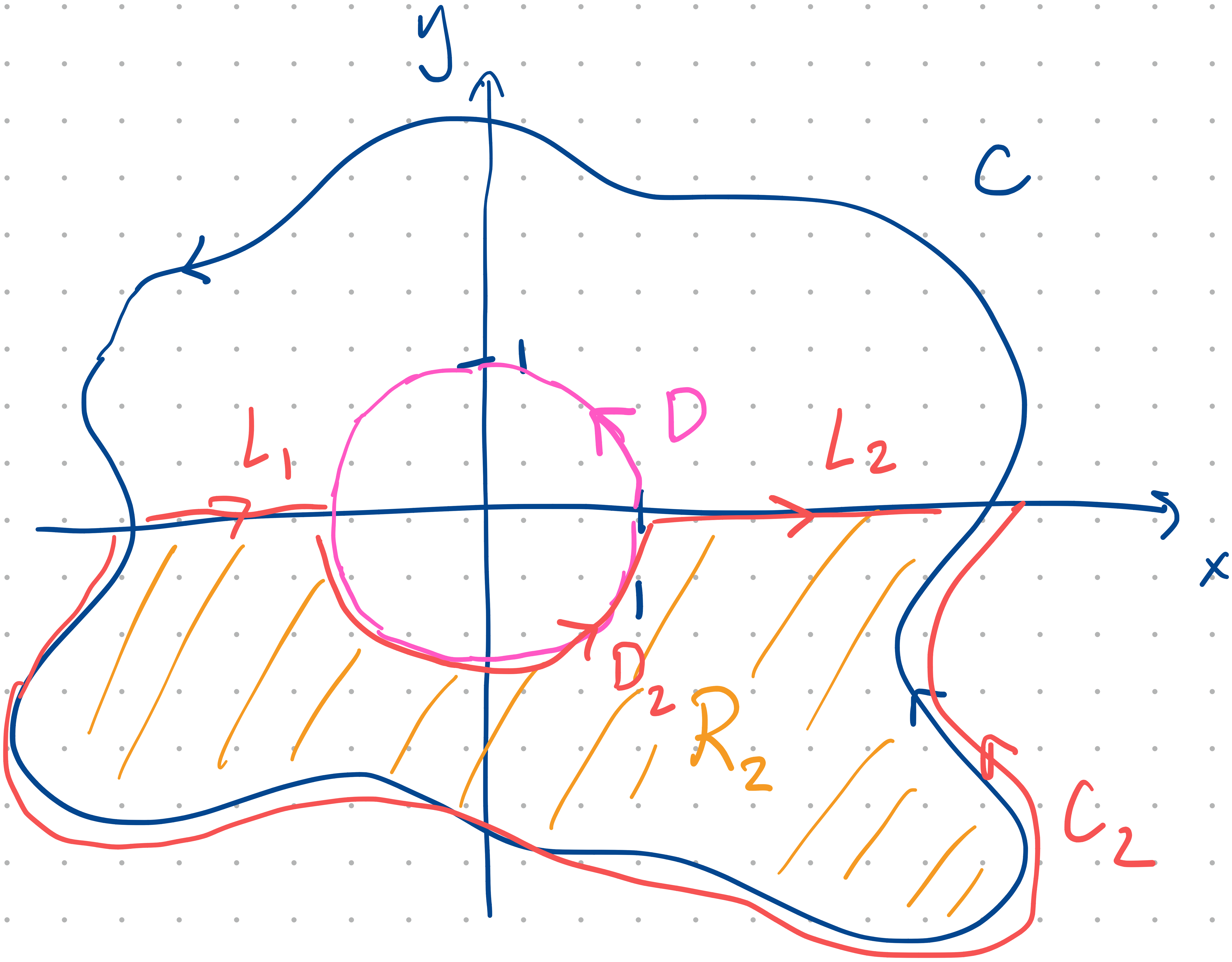
enclosed in C.



I claim that $\oint_C \vec{F}\cdot d\vec{r} = \oint_D \vec{F}\cdot d\vec{r}$.

(Refer to Example 5 in 16.4 and the discussion preceding)


y
C
C_1
D_1
R_1
D
L_1
L_2
x


y
C
D
L_1
L_2
x
D_2
R_2
C_2

Apply Green's Thm to R_1:

$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{L_1} \vec{F}\cdot d\vec{r} + \oint_{-D_1} \vec{F}\cdot d\vec{r} + \oint_{L_2} \vec{F}\cdot d\vec{r}$$

$$= \iint_{R_1} (Q_x - P_y)\, dA = 0$$

To R_2:

$$\oint_{C_2} \vec{F}\cdot d\vec{r} + \oint_{-L_2} \vec{F}\cdot d\vec{r} + \oint_{-D_2} \vec{F}\cdot d\vec{r} + \oint_{-L_1} \vec{F}\cdot d\vec{r}$$

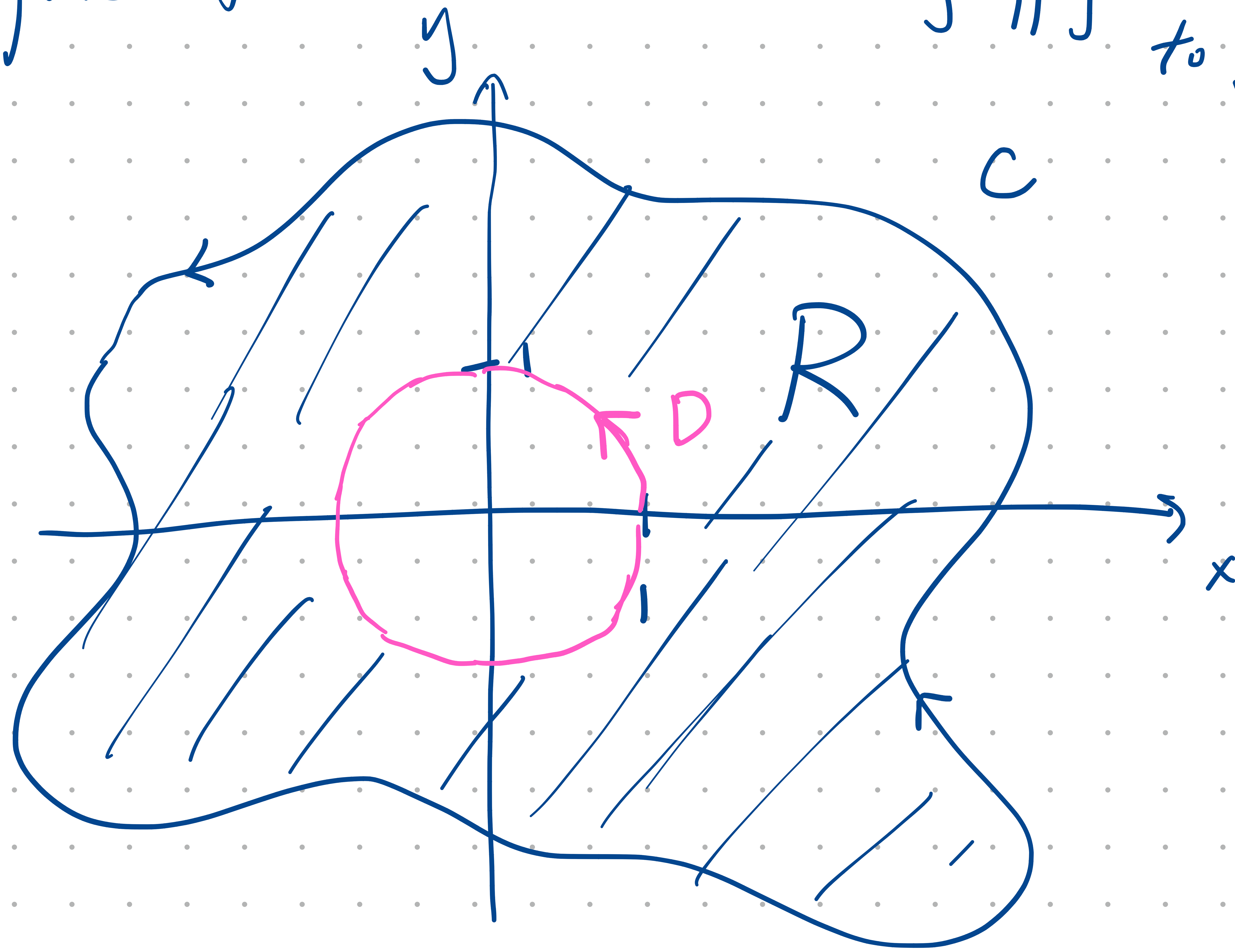
$$= \iint_{R_2} (Q_x - P_y)\, dA = 0.$$

Add:

$$\oint_{C} \vec{F}\cdot d\vec{r} + \oint_{-D} \vec{F}\cdot d\vec{r} = 0$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_D \vec{F}\cdot d\vec{r} \quad \text{as desired}$$

If you're comfortable w/ it, alternatively apply Green's to R



"positively oriented boundary of R" =

$$C + (-D)$$

← reverse of D

by RHR

so Green's Thm gives $\oint_C \vec{F}\cdot d\vec{r} + \oint_{-D} \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\,dA$

$= 0$

as before.

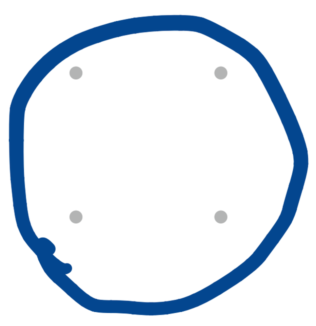
Now we just need to compute

$$\oint_D \left\langle \frac{2x^3+2xy^2-2y}{x^2+y^2}, \frac{2y^3+2x^2y+2x}{x^2+y^2} \right\rangle \cdot d\vec{r}$$

D

$x^2+y^2=1$

CCW



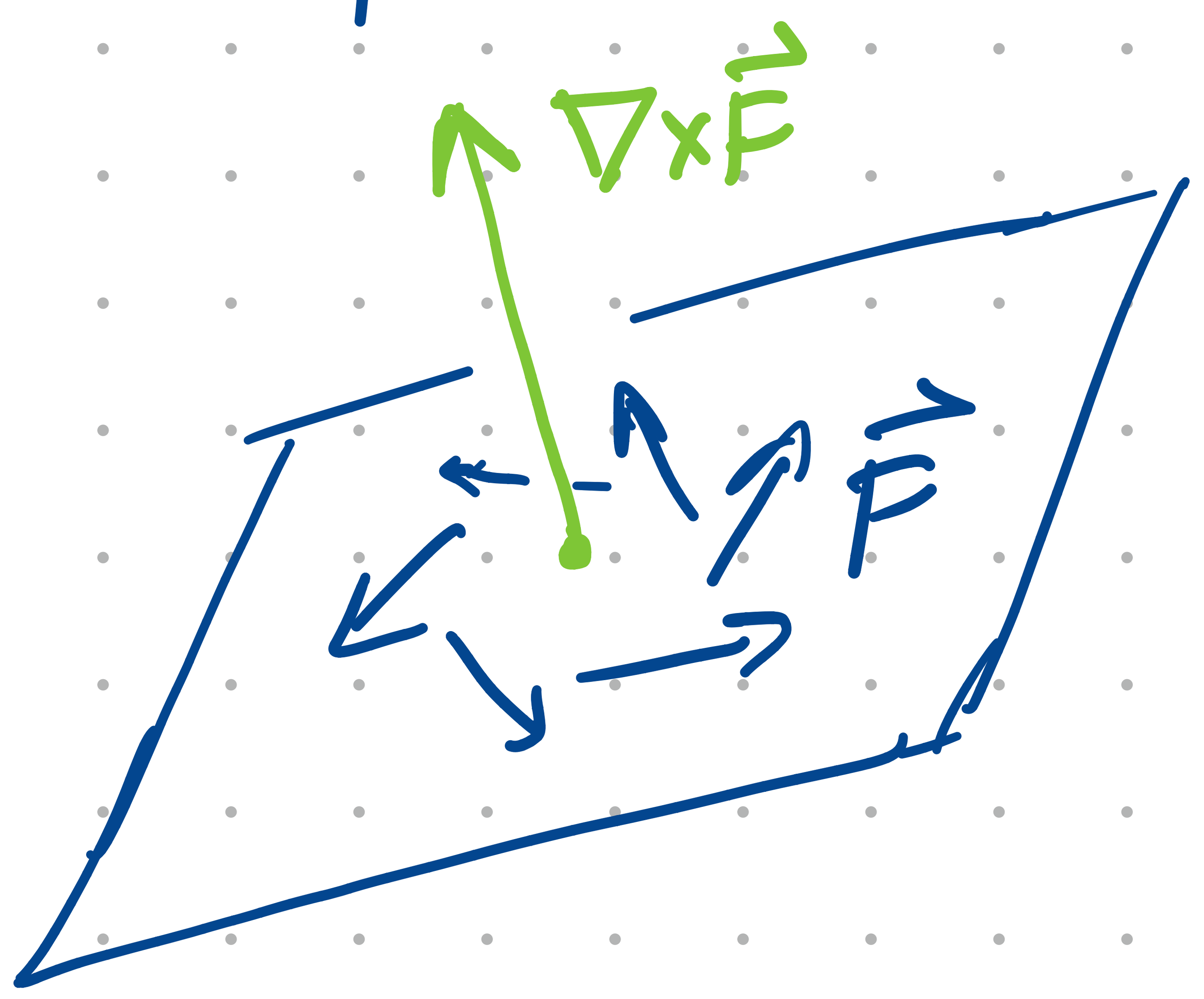
$$= \oint_D \langle 2x^3+2xy^2-2y,\ 2y^3+2x^2y+2x\rangle \cdot d\vec{r}$$

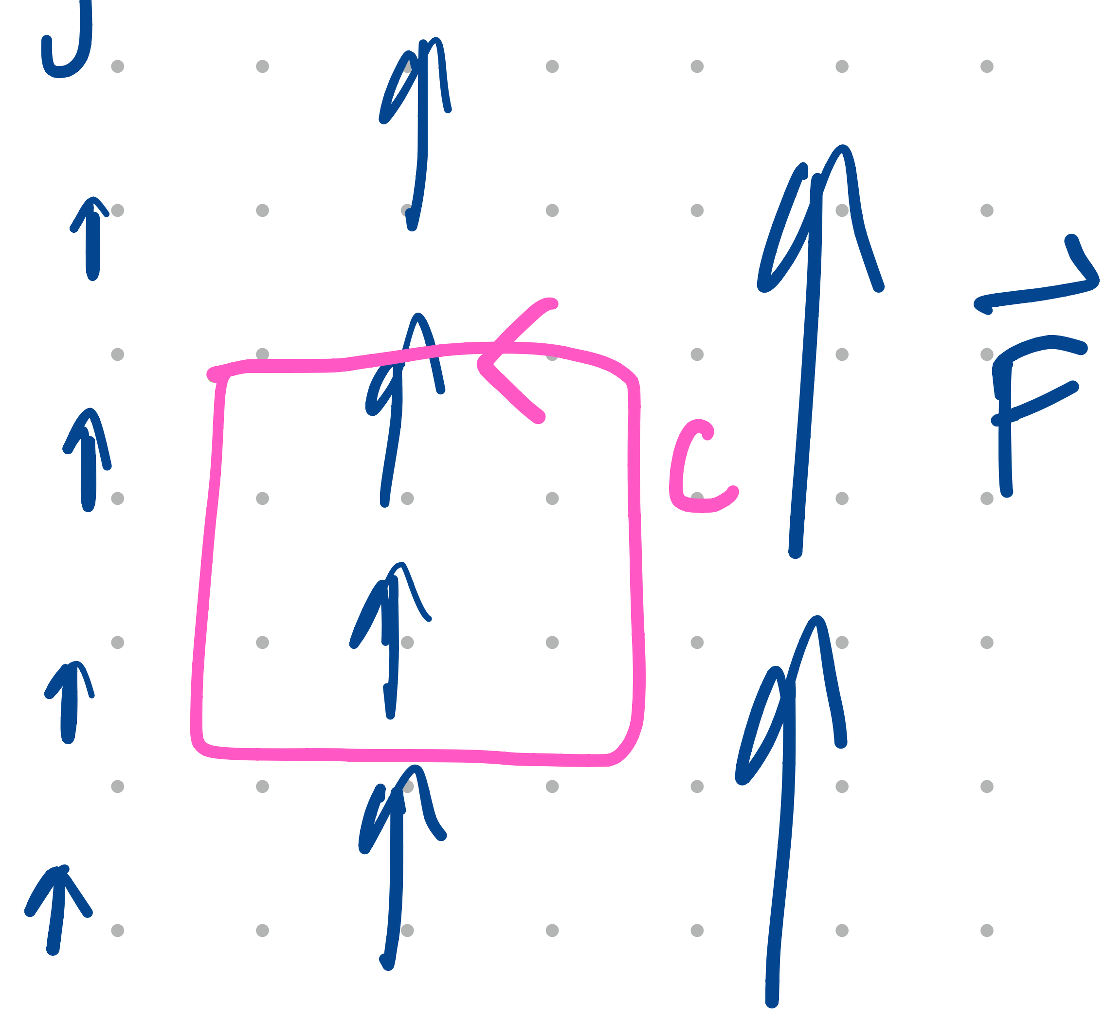
I can use Green's Thm on this now!

$$= \iint_{x^2+y^2\le 1} \left((4xy+2) - (4xy-2) \right) dx\,dy$$

$$= \iint_{x^2+y^2\le 1} 4 \; dx\,dy = 4 \text{ Area} = \boxed{4\pi}$$

Loose interpretation of "curl" is by RHR:





$\nabla\times\vec{F}$ points out of the page

in 2D: $\left|\nabla\times\vec{F}\right| "=" \lim\limits_{\square \text{ shrinks}} \dfrac{\oint_C \vec{F}\cdot d\vec{r}}{\text{Area}(\square)}$

Another interpretation: imagine $\vec{F}$ describes water flow, and let an object float on the water surface. "Curl F" measures how the object spins.

How does $\nabla\times$ relate to Clairaut's Thm?

Clairaut's Thm: $f(x,y,z)$ a "nice" fn... then

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}.$$

Corollary: $f(x,y,z)$ "nice" then $\nabla\times(\nabla f) = 0$.

pf: $\nabla f = \langle f_x, f_y, f_z \rangle$

$$\nabla\times(\nabla f) = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{bmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

$$= \langle 0, 0, 0 \rangle$$ by Clairaut's thm. $\square$

Alternative writing: $\nabla f = \langle P, Q, R \rangle$

$$\begin{array}{ccc} P & Q & R \\ \| & \| & \| \\ f_x & f_y & f_z \end{array}$$

Clairaut's says $Q_x - P_y = 0$ etc...

$$\begin{array}{cc} Q_x & P_y \\ \| & \| \\ f_{yx} & f_{xy} \end{array}$$